Explicit Port-Hamiltonian Representation of Feedthrough-Systems with Nonlinear Dissipation

Explizite Port-Hamiltonsche Darstellung von Durchgriffsystemen mit nichtlinearer Dissipation

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Abstract — In this technical note, we present an explicit port-Hamiltonian formulation of feedthrough-systems subject to nonlinear energy-dissipating effects. To this end, we merge the Dirac structure—which describes the system's internal interconnection structure—with the constitutive relations of energy-storing and energy-dissipating elements. The resulting port-Hamiltonian system (PHS) is proven to be passive and generalizes an existing nonlinear-dissipative port-Hamiltonian formulation from the literature by feedthrough.

Zusammenfassung — In diesem Beitrag wird eine explizite Port-Hamiltonsche Formulierung von Durchgriffsystemen mit nichtlinear-dissipativen Effekten vorgestellt. Die Herleitung der Systemgleichungen erfolgt durch die Zusammenführung der Dirac-Struktur des Systems und den Konstitutiv-Gleichungen der energiespeichernden und energiedissipierenden Systemelemente. Die Passivität des vorgestellten Port-Hamiltonschen Systems (PHS) wird mathematisch nachgewiesen. Ferner zeigt sich, dass die Systemformulierung eine Verallgemeinerung eines aus der Literatur bekannten nichtlinear-dissipativen PHS darstellt.

I. Introduction

Port-Hamiltonian systems (PHSs) are a powerful framework for developing control systems for complex physical systems. PHSs have first been introduced for real-valued, continuous-time nonlinear systems with lumped parameters (see, e.g., [1, 2]). Meanwhile, the port-Hamiltonian framework has been extended to complex-valued systems (see, e.g., [3]), discrete-time systems (see, e.g., [4]), and distributed-parameter systems (see, e.g., [5, 6]).

Port-Hamiltonian methods have three advantages over standard state-space approaches: Firstly, they are based on energy as domain-independent conserved quantity. This enables to treat multi-domain systems in a unifying methodological framework. Secondly, PHSs are passive in consequence of their system formulation. Therewith, they provide an ideal basis for the powerful methods from passivity-based nonlinear control. Thirdly, port-Hamiltonian methods are highly modular and scalable to large systems. Due to these three reasons, PHSs are of great interest when developing control systems for nonlinear mechatronic systems, see [7, pp. 131ff.].

Port-Hamiltonian control design methods are model-based. The majority of methods is based on an explicit port-Hamiltonian model, i.e., a PHS in form of an ordinary differential equation (ODE), see, e.g., [8], [9], [10]. For systems with energy-dissipating elements that are linear (e.g., Ohm's law), there exist well known explicit port-Hamiltonian representations. This applies to both, systems with and without feedthrough, see, e.g., [7, pp. 70–71]. For the case of nonlinear energy-dissipation (e.g., systems with nonlinear friction), the situation is different. Indeed, the author of [9, p. 114] proposes a port-Hamiltonian representation for such systems without feedthrough. However, as of now, there has been no reports on a port-Hamiltonian ODE representation of systems with nonlinear energy dissipation and feedthrough.

In this contribution, we bridge this research gap. We propose an explicit port-Hamiltonian representation for systems with nonlinear energy-dissipating effects and feedthrough. Passivity of the port-Hamiltonian representation is proven. Moreover, we show that our representation is a generalization of the PHS proposed in [9, p. 114].

Notation: Sets and spaces are written in blackboard bold. The set $\mathbb{R}$ is the set of real numbers. Vectors and matrices are written in bold font. Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ be a matrix with m rows and n columns. For the transpose of $\boldsymbol{A}$ we write $\boldsymbol{A}^\top$. Consider a continuously differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}$, $\boldsymbol{x} \mapsto f(\boldsymbol{x})$. We call f non-negative if $f(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

Throughout this paper, the time-dependence "(t)" of vectors is omitted in the notation.

II. Problem Statement

We consider a physical system being composed of N_C energy-storing elements, N_R energy-dissipating elements, and N_P energy sources.[1] The total number of system elements is given by $N_\mathrm{E} = N_\mathrm{C} + N_\mathrm{R} + N_\mathrm{P}$. Each system element is equipped with a so-called power port to interact with the other system elements. A power port is de-

[1] Energy sinks can be seen as negative energy sources.

scribed by an input variable u_i and an output variable y_i, $i = 1, \ldots, N_\mathrm{E}$.

Fig. 1 depicts an exemplary system with one energy-storing element C_1, two energy-dissipating elements R_1 and R_2, and one source of energy P_1. The half arrows—so-called *bonds* [7, pp. 4ff.]—symbolize an exchange of power through the elements' ports. The direction of a bond determines the positive direction of the associated power flow.

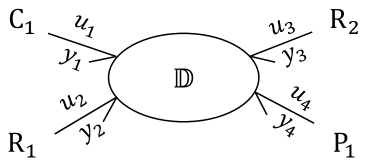

Fig. 1. Illustrative example system with $N_\mathrm{C} = 1$, $N_\mathrm{R} = 2$, and $N_\mathrm{P} = 1$

The input variables of the storage elements, dissipating elements, and source elements are collected in the vectors $\boldsymbol{u}_\mathrm{C} \in \mathbb{R}^{N_\mathrm{C}}$, $\boldsymbol{u}_\mathrm{R} \in \mathbb{R}^{N_\mathrm{R}}$, and $\boldsymbol{u}_\mathrm{P} \in \mathbb{R}^{N_\mathrm{P}}$, respectively; the output variables are summarized in the vectors $\boldsymbol{y}_\mathrm{C} \in \mathbb{R}^{N_\mathrm{C}}$, $\boldsymbol{y}_\mathrm{R} \in \mathbb{R}^{N_\mathrm{R}}$, and $\boldsymbol{y}_\mathrm{P} \in \mathbb{R}^{N_\mathrm{P}}$, respectively.

Let the constitutive relations of the energy-storing elements be given as:

$$\boldsymbol{y}_\mathrm{C} = -\dot{\boldsymbol{x}}, \qquad \boldsymbol{u}_\mathrm{C} = \frac{\partial H}{\partial \boldsymbol{x}}(\boldsymbol{x}), \tag{1}$$

where $\boldsymbol{x} \in \mathbb{X} \subseteq \mathbb{R}^{N_\mathrm{C}}$ is the energy state and $H \colon \mathbb{X} \to \mathbb{R}$ is a differentiable storage function that is bounded from below. For the energy-dissipating elements, we suppose nonlinear constitutive relations which are expressed as the graph of an input-output map:

$$\boldsymbol{u}_\mathrm{R} = \boldsymbol{\Phi}(\boldsymbol{y}_\mathrm{R}, \boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}_\mathrm{P}), \tag{2}$$

where $\boldsymbol{y}_\mathrm{R}^\top \boldsymbol{u}_\mathrm{R} \leq 0$. In (2), $\boldsymbol{z} := \frac{\partial H}{\partial \boldsymbol{x}}(\boldsymbol{x})$ is the co-state of the system.

The interconnection of the system elements is described by a modulated Dirac structure, see Fig. 1. A Dirac structure is a power-conserving, geometric structure which describes the interconnection between the system elements. A detailed introduction into the concept of Dirac structures can be found in [7] and [11]. We suppose a Dirac structure in input-output representation [7, p. 87] in which the inputs are mapped to the outputs:

$$\mathbb{D}(\boldsymbol{x}) = \{ (\begin{pmatrix} \boldsymbol{u}_\mathrm{C} \\ \boldsymbol{u}_\mathrm{R} \\ \boldsymbol{u}_\mathrm{P} \end{pmatrix}, \begin{pmatrix} \boldsymbol{y}_\mathrm{C} \\ \boldsymbol{y}_\mathrm{R} \\ \boldsymbol{y}_\mathrm{P} \end{pmatrix}) \in \mathbb{R}^{N_\mathrm{E}} \times \mathbb{R}^{N_\mathrm{E}} \mid$$
$$\begin{pmatrix} \boldsymbol{y}_\mathrm{C} \\ \boldsymbol{y}_\mathrm{R} \\ \boldsymbol{y}_\mathrm{P} \end{pmatrix} = \underbrace{\begin{pmatrix} \boldsymbol{Z}_\mathrm{CC}(\boldsymbol{x}) & -\boldsymbol{Z}_\mathrm{CR}(\boldsymbol{x}) & -\boldsymbol{Z}_\mathrm{CP}(\boldsymbol{x}) \\ \boldsymbol{Z}_\mathrm{CR}^\top(\boldsymbol{x}) & \boldsymbol{Z}_\mathrm{RR}(\boldsymbol{x}) & -\boldsymbol{Z}_\mathrm{RP}(\boldsymbol{x}) \\ \boldsymbol{Z}_\mathrm{CP}^\top(\boldsymbol{x}) & \boldsymbol{Z}_\mathrm{RP}^\top(\boldsymbol{x}) & \boldsymbol{Z}_\mathrm{PP}(\boldsymbol{x}) \end{pmatrix}}_{\boldsymbol{Z}(\boldsymbol{x})} \begin{pmatrix} \boldsymbol{u}_\mathrm{C} \\ \boldsymbol{u}_\mathrm{R} \\ \boldsymbol{u}_\mathrm{P} \end{pmatrix} \}, \tag{3}$$

where $\boldsymbol{Z}(\boldsymbol{x}) = -\boldsymbol{Z}^\top(\boldsymbol{x}) \in \mathbb{R}^{N_\mathrm{E} \times N_\mathrm{E}}$ for all $\boldsymbol{x} \in \mathbb{X}$.

The problem to be addressed in this paper now reads:

Problem 2.1. *Given a system described by* (1), (2), *and* (3). *What is an explicit port-Hamiltonian formulation of this system in case of feedthrough?*

Remark 2.2. *The input and output variables usually represent generalized efforts and flows. These generalized power variables allow for correspondences in various physical domains, see, e.g., [7, p. 23]. For example, in the electrical domain, the effort corresponds to a voltage and the flow corresponds to a current; in the mechanical domain the effort and flow may be related to a velocity and a force, respectively. Therewith, the product between input and output variables has the unit of power.*

Remark 2.3. *Equations* (1) *and* (2) *are well-known standard representations for the constitutive relations of nonlinear energy-storing and energy-dissipating elements, see [12, p. 357] and [11, p. 24], respectively. Moreover, for many systems, a Dirac structure in the form* (3) *can be computed in an automated manner, see [13, 14].*

Remark 2.4. *The power balance of* (3) *is*

$$\begin{pmatrix} \boldsymbol{u}_\mathrm{C}^\top & \boldsymbol{u}_\mathrm{R}^\top & \boldsymbol{u}_\mathrm{P}^\top \end{pmatrix} \begin{pmatrix} \boldsymbol{y}_\mathrm{C} \\ \boldsymbol{y}_\mathrm{R} \\ \boldsymbol{y}_\mathrm{P} \end{pmatrix} = \begin{pmatrix} \boldsymbol{u}_\mathrm{C}^\top & \boldsymbol{u}_\mathrm{R}^\top & \boldsymbol{u}_\mathrm{P}^\top \end{pmatrix} \boldsymbol{Z}(\boldsymbol{x}) \begin{pmatrix} \boldsymbol{u}_\mathrm{C} \\ \boldsymbol{u}_\mathrm{R} \\ \boldsymbol{u}_\mathrm{P} \end{pmatrix} = 0, \tag{4}$$

where the last equality follows from the skew-symmetry of $\boldsymbol{Z}(\boldsymbol{x})$. *Equation* (4) *shows that the total power entering the Dirac structure is zero, i.e., the power-conservation of the Dirac structure.*

In the following section, we present and prove our main results regarding Problem 2.1.

III. Main Results

Before we state the main results of this paper, we make the following assumption to exclude interdependencies between energy-dissipating elements:

Assumption 3.1. *In* (3), *we have* $\boldsymbol{Z}_\mathrm{RR}(\boldsymbol{x}) = \boldsymbol{0}$ *for all* $\boldsymbol{x} \in \mathbb{X}$.

Now for the first main result of this paper:

Proposition 3.2. *Consider a system with constitutive relations of storage elements and dissipative elements in the forms* (1) *and* (2), *respectively. Moreover, let the interconnection structure of the system be given as a Dirac structure* (3) *which satisfies Assumption 3.1.*

Equations (1), (2), *and* (3) *can be written as an explicit input-state-output PHS of the form*

$$\dot{\boldsymbol{x}} = \boldsymbol{J}(\boldsymbol{x})\,\boldsymbol{z} - \boldsymbol{\mathcal{R}}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) + \boldsymbol{G}(\boldsymbol{x})\,\boldsymbol{u}, \tag{5a}$$
$$\boldsymbol{y} = \boldsymbol{G}^\top(\boldsymbol{x})\,\boldsymbol{z} + \boldsymbol{\mathcal{P}}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) + \boldsymbol{M}(\boldsymbol{x})\,\boldsymbol{u}, \tag{5b}$$

with vectors $\boldsymbol{x} \in \mathbb{X} \subseteq \mathbb{R}^n$, $\boldsymbol{z} \in \mathbb{Z} \subseteq \mathbb{R}^n$, $\boldsymbol{u} \in \mathbb{U} \subseteq \mathbb{R}^p$, *mappings* $\boldsymbol{\mathcal{R}}(\boldsymbol{x}, \cdot, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$, $\boldsymbol{\mathcal{P}}(\boldsymbol{x}, \cdot, \cdot) : \mathbb{R}^n \to \mathbb{R}^p$, *and* $\boldsymbol{u} = \boldsymbol{u}_\mathrm{P}$, $\boldsymbol{y} = \boldsymbol{y}_\mathrm{P}$.[2] *The dimensions* n *and* p *are given as* $n = N_\mathrm{C}$ *and* $p = N_\mathrm{P}$. *In* (5), *the matrices and mappings satisfy* $\boldsymbol{J}(\boldsymbol{x}) = -\boldsymbol{J}^\top(\boldsymbol{x})$, $\boldsymbol{M}(\boldsymbol{x}) = -\boldsymbol{M}^\top(\boldsymbol{x})$, *and*

$$\begin{pmatrix} \boldsymbol{z} \\ \boldsymbol{u} \end{pmatrix}^\top \begin{pmatrix} \boldsymbol{\mathcal{R}}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) & \boldsymbol{0} \\ 0 & \boldsymbol{\mathcal{P}}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \end{pmatrix} \geq 0, \tag{6}$$

[2] By $\mathbb{Z}$ we denote the real-valued co-state-space. In particular, we do not refer to the set of integers.

for all $\boldsymbol{x} \in \mathbb{X}$, $\boldsymbol{z} \in \mathbb{Z}$, and $\boldsymbol{u} \in \mathbb{U}$. The matrices can be obtained from $\boldsymbol{J}(\boldsymbol{x}) = -\boldsymbol{Z}_{\mathrm{CC}}(\boldsymbol{x})$, $\boldsymbol{G}(\boldsymbol{x}) = \boldsymbol{Z}_{\mathrm{CP}}(\boldsymbol{x})$, and $\boldsymbol{M}(\boldsymbol{x}) = \boldsymbol{Z}_{\mathrm{PP}}(\boldsymbol{x})$; the mappings are calculated as

$$\boldsymbol{\mathcal{R}}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) = -\boldsymbol{Z}_{\mathrm{CR}}(\boldsymbol{x})\,\boldsymbol{\Phi}\big(\boldsymbol{Z}_{\mathrm{CR}}^{\top}(\boldsymbol{x})\,\boldsymbol{z} - \boldsymbol{Z}_{\mathrm{CP}}(\boldsymbol{x})\,\boldsymbol{u},\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}\big), \tag{7a}$$

$$\boldsymbol{\mathcal{P}}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) = \boldsymbol{Z}_{\mathrm{RP}}^{\top}(\boldsymbol{x})\,\boldsymbol{\Phi}\big(\boldsymbol{Z}_{\mathrm{CR}}^{\top}(\boldsymbol{x})\,\boldsymbol{z} - \boldsymbol{Z}_{\mathrm{CP}}(\boldsymbol{x})\,\boldsymbol{u},\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}\big). \tag{7b}$$

Proof. Inserting $\boldsymbol{u}_{\mathrm{C}} = \boldsymbol{z}$ and (2) into the first line of the equation system from (3) yields

$$\boldsymbol{y}_{\mathrm{C}} = \boldsymbol{Z}_{\mathrm{CC}}(\boldsymbol{x})\,\boldsymbol{u}_{\mathrm{C}} - \boldsymbol{Z}_{\mathrm{CR}}(\boldsymbol{x})\,\boldsymbol{\Phi}(\boldsymbol{y}_{\mathrm{R}},\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}_{\mathrm{P}}) - \boldsymbol{Z}_{\mathrm{CP}}(\boldsymbol{x})\,\boldsymbol{u}_{\mathrm{P}}. \tag{8}$$

For the second term of the right side we write

$$\begin{aligned} &-\boldsymbol{Z}_{\mathrm{CR}}(\boldsymbol{x})\,\boldsymbol{\Phi}(\boldsymbol{y}_{\mathrm{R}},\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}_{\mathrm{P}}) \\ &\overset{(2)}{=} \underbrace{-\boldsymbol{Z}_{\mathrm{CR}}(\boldsymbol{x})\,\boldsymbol{\Phi}\left(\boldsymbol{Z}_{\mathrm{CR}}^{\top}(\boldsymbol{x})\,\boldsymbol{z} - \boldsymbol{Z}_{\mathrm{CP}}(\boldsymbol{x})\,\boldsymbol{u}_{\mathrm{P}},\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}_{\mathrm{P}}\right)}_{=\boldsymbol{\mathcal{R}}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}_{\mathrm{P}})}. \end{aligned} \tag{9}$$

By inserting (9) into (8) we obtain (5a):

$$\begin{aligned} \boldsymbol{y}_{\mathrm{C}} &= \boldsymbol{Z}_{\mathrm{CC}}(\boldsymbol{x})\,\boldsymbol{z} + \boldsymbol{\mathcal{R}}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}_{\mathrm{P}}) - \boldsymbol{Z}_{\mathrm{CP}}(\boldsymbol{x})\,\boldsymbol{u}_{\mathrm{P}} \\ \overset{(1)}{\Leftrightarrow} -\dot{\boldsymbol{x}} &= \boldsymbol{Z}_{\mathrm{CC}}(\boldsymbol{x})\,\boldsymbol{z} + \boldsymbol{\mathcal{R}}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}_{\mathrm{P}}) - \boldsymbol{Z}_{\mathrm{CP}}(\boldsymbol{x})\,\boldsymbol{u}_{\mathrm{P}} \\ \Leftrightarrow \quad \dot{\boldsymbol{x}} &= \underbrace{-\boldsymbol{Z}_{\mathrm{CC}}(\boldsymbol{x})}_{=\boldsymbol{J}(\boldsymbol{x})}\boldsymbol{z} - \boldsymbol{\mathcal{R}}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}_{\mathrm{P}}) + \underbrace{\boldsymbol{Z}_{\mathrm{CP}}(\boldsymbol{x})}_{=\boldsymbol{G}(\boldsymbol{x})}\underbrace{\boldsymbol{u}_{\mathrm{P}}}_{=\boldsymbol{u}}. \end{aligned} \tag{10}$$

Now for the output equation. From the third line of the equation system in (3) and with $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{P}}$, $\boldsymbol{y} = \boldsymbol{y}_{\mathrm{P}}$ we obtain

$$\boldsymbol{y} = \underbrace{\boldsymbol{Z}_{\mathrm{CP}}^{\top}(\boldsymbol{x})}_{=\boldsymbol{G}^{\top}(\boldsymbol{x})}\boldsymbol{z} + \boldsymbol{Z}_{\mathrm{RP}}^{\top}(\boldsymbol{x})\,\boldsymbol{u}_{\mathrm{R}} + \underbrace{\boldsymbol{Z}_{\mathrm{PP}}^{\top}(\boldsymbol{x})}_{=\boldsymbol{M}(\boldsymbol{x})}\boldsymbol{u}. \tag{11}$$

For the second term from the right side we write

$$\begin{aligned} \boldsymbol{Z}_{\mathrm{RP}}^{\top}(\boldsymbol{x})\,\boldsymbol{u}_{\mathrm{R}} &\overset{(2)}{=} \boldsymbol{Z}_{\mathrm{RP}}^{\top}(\boldsymbol{x})\,\boldsymbol{\Phi}(\boldsymbol{y}_{\mathrm{R}},\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) \\ &\overset{(3)}{=} \underbrace{\boldsymbol{Z}_{\mathrm{RP}}^{\top}(\boldsymbol{x})\,\boldsymbol{\Phi}\left(\boldsymbol{Z}_{\mathrm{CR}}^{\top}(\boldsymbol{x})\,\boldsymbol{z} - \boldsymbol{Z}_{\mathrm{CP}}(\boldsymbol{x})\,\boldsymbol{u},\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}\right)}_{=\boldsymbol{\mathcal{P}}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u})}. \end{aligned} \tag{12}$$

Inserting (12) into (11) yields (5b).

Next, we show that (6) holds. By multiplying (5a) from the right side with $\boldsymbol{z}^{\top}$ we obtain

$$\boldsymbol{z}^{\top}\dot{\boldsymbol{x}} = -\boldsymbol{z}^{\top}\boldsymbol{\mathcal{R}}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) + \boldsymbol{z}^{\top}\boldsymbol{G}(\boldsymbol{x})\,\boldsymbol{u}. \tag{13}$$

On the other hand, by the power balance (4) of the Dirac structure we have

$$\boldsymbol{z}^{\top}\dot{\boldsymbol{x}} \overset{(1)}{=} -\boldsymbol{u}_{\mathrm{C}}^{\top}\boldsymbol{y}_{\mathrm{C}} \overset{(4)}{=} \boldsymbol{u}_{\mathrm{R}}^{\top}\boldsymbol{y}_{\mathrm{R}} + \boldsymbol{u}_{\mathrm{P}}^{\top}\boldsymbol{y}_{\mathrm{P}}. \tag{14}$$

Equating (13) and (14) yields

$$-\boldsymbol{z}^{\top}\boldsymbol{\mathcal{R}}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) + \boldsymbol{z}^{\top}\boldsymbol{G}(\boldsymbol{x})\,\boldsymbol{u} = \boldsymbol{u}_{\mathrm{R}}^{\top}\boldsymbol{y}_{\mathrm{R}} + \boldsymbol{u}_{\mathrm{P}}^{\top}\boldsymbol{y}_{\mathrm{P}}. \tag{15}$$

The last term reads

$$\begin{aligned} \boldsymbol{u}_{\mathrm{P}}^{\top}\boldsymbol{y}_{\mathrm{P}} &\overset{(3)}{=} \boldsymbol{u}_{\mathrm{P}}^{\top}\left(\boldsymbol{Z}_{\mathrm{CP}}^{\top}(\boldsymbol{x})\,\boldsymbol{u}_{\mathrm{C}} + \boldsymbol{Z}_{\mathrm{RP}}^{\top}(\boldsymbol{x})\,\boldsymbol{u}_{\mathrm{R}} + \boldsymbol{Z}_{\mathrm{PP}}(\boldsymbol{x})\,\boldsymbol{u}_{\mathrm{P}}\right) \\ &= \boldsymbol{u}_{\mathrm{P}}^{\top}\boldsymbol{Z}_{\mathrm{CP}}^{\top}(\boldsymbol{x})\,\boldsymbol{u}_{\mathrm{C}} + \boldsymbol{u}_{\mathrm{P}}^{\top}\boldsymbol{Z}_{\mathrm{RP}}^{\top}(\boldsymbol{x})\,\boldsymbol{u}_{\mathrm{R}} \\ &\overset{(2)}{=} \boldsymbol{u}_{\mathrm{P}}^{\top}\boldsymbol{Z}_{\mathrm{CP}}^{\top}(\boldsymbol{x})\,\boldsymbol{u}_{\mathrm{C}} + \boldsymbol{u}_{\mathrm{P}}^{\top}\boldsymbol{Z}_{\mathrm{RP}}^{\top}(\boldsymbol{x})\,\boldsymbol{\Phi}(\boldsymbol{y}_{\mathrm{R}},\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) \\ &\overset{(3)}{=} \boldsymbol{u}_{\mathrm{P}}^{\top}\boldsymbol{Z}_{\mathrm{CP}}^{\top}(\boldsymbol{x})\,\boldsymbol{u}_{\mathrm{C}} \\ &\quad + \boldsymbol{u}_{\mathrm{P}}^{\top}\boldsymbol{Z}_{\mathrm{RP}}^{\top}(\boldsymbol{x})\,\boldsymbol{\Phi}\left(\boldsymbol{Z}_{\mathrm{CR}}^{\top}(\boldsymbol{x})\,\boldsymbol{z} - \boldsymbol{Z}_{\mathrm{CP}}(\boldsymbol{x})\,\boldsymbol{u},\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}\right). \end{aligned} \tag{16}$$

With (7a) and $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{P}}$, $\boldsymbol{y} = \boldsymbol{y}_{\mathrm{P}}$ we write (16) as

$$\boldsymbol{u}_{\mathrm{P}}^{\top}\boldsymbol{y}_{\mathrm{P}} = \boldsymbol{u}^{\top}\underbrace{\boldsymbol{Z}_{\mathrm{CP}}^{\top}(\boldsymbol{x})}_{=\boldsymbol{G}^{\top}(\boldsymbol{x})}\boldsymbol{z} + \boldsymbol{u}^{\top}\mathcal{P}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}). \tag{17}$$

By inserting (17) into (15) we prove (6):

$$-\boldsymbol{z}^{\top}\boldsymbol{\mathcal{R}}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) - \boldsymbol{u}^{\top}\mathcal{P}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) = \boldsymbol{u}_{\mathrm{R}}^{\top}\boldsymbol{y}_{\mathrm{R}} \overset{(2)}{\leq} 0. \tag{18}$$

□

To the best of our knowledge, the explicit PHS (5) has not been presented in the literature so far. The next proposition shows that this PHS is passive.

Proposition 3.3. *The PHS (5) is passive.*

Proof. Recall $\boldsymbol{z} = \frac{\partial H}{\partial \boldsymbol{x}}(\boldsymbol{x})$ with $H(\boldsymbol{x})$ as a storage function that is bounded from below. We always find a constant $c \in \mathbb{R}_{\geq 0}$ such that $\tilde{H}(\boldsymbol{x}) = H(\boldsymbol{x}) + c$ is a non-negative function. The time derivative of $\tilde{H}(\boldsymbol{x})$ reads

$$\begin{aligned} \dot{\tilde{H}}(\boldsymbol{x}) &= \left(\frac{\partial \tilde{H}}{\partial \boldsymbol{x}}(\boldsymbol{x})\right)^{\top}\dot{\boldsymbol{x}} = \left(\frac{\partial H}{\partial \boldsymbol{x}}(\boldsymbol{x})\right)^{\top}\dot{\boldsymbol{x}} \\ &= \boldsymbol{z}^{\top}\left(\boldsymbol{J}(\boldsymbol{x})\,\boldsymbol{z} - \boldsymbol{\mathcal{R}}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) + \boldsymbol{G}(\boldsymbol{x})\,\boldsymbol{u}\right) \\ &= -\boldsymbol{z}^{\top}\boldsymbol{\mathcal{R}}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) + \boldsymbol{z}^{\top}\boldsymbol{G}(\boldsymbol{x})\,\boldsymbol{u}. \end{aligned} \tag{19}$$

Transposing (5b) and multiplying with $\boldsymbol{u}$ from the right gives

$$\begin{aligned} \boldsymbol{y}^{\top}\boldsymbol{u} &= \boldsymbol{z}^{\top}\boldsymbol{G}(\boldsymbol{x})\,\boldsymbol{u} + \boldsymbol{\mathcal{P}}^{\top}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u})\,\boldsymbol{u} \\ \Leftrightarrow \quad \boldsymbol{z}^{\top}\boldsymbol{G}(\boldsymbol{x})\,\boldsymbol{u} &= \boldsymbol{y}^{\top}\boldsymbol{u} - \boldsymbol{u}^{\top}\boldsymbol{\mathcal{P}}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}). \end{aligned} \tag{20}$$

Inserting (20) into (19) then yields

$$\begin{aligned} \dot{\tilde{H}}(\boldsymbol{x}) &= -\boldsymbol{z}^{\top}\boldsymbol{\mathcal{R}}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) + \boldsymbol{y}^{\top}\boldsymbol{u} - \boldsymbol{u}^{\top}\boldsymbol{\mathcal{P}}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) \\ &= \boldsymbol{y}^{\top}\boldsymbol{u} - \underbrace{\begin{pmatrix}\boldsymbol{z}\\ \boldsymbol{u}\end{pmatrix}^{\top}\begin{pmatrix}\boldsymbol{\mathcal{R}}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) & \boldsymbol{0}\\ 0 & \boldsymbol{\mathcal{P}}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u})\end{pmatrix}}_{\geq 0} \leq \boldsymbol{y}^{\top}\boldsymbol{u}. \end{aligned} \tag{21}$$

□

For the case of no feedthrough, we obtain from Proposition 3.2 the "input-state-output PHS with nonlinear resistive structure" introduced by [9, Def. 6.1.4]. This special case is outlined in the subsequent corollary.

Corollary 3.4. *Given an explicit Dirac structure (3) which satisfies Assumption 3.1. Let $\boldsymbol{Z}_{\mathrm{RP}}(\boldsymbol{x}) = 0$ and $\boldsymbol{Z}_{\mathrm{PP}}(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in \mathbb{X}$. Equations (1), (2), and (3) can be written as an explicit input-state-output PHS of the form*

$$\dot{\boldsymbol{x}} = \boldsymbol{J}(\boldsymbol{x})\,\boldsymbol{z} - \boldsymbol{\mathcal{R}}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) + \boldsymbol{G}(\boldsymbol{x})\,\boldsymbol{u}, \tag{22a}$$

$$\boldsymbol{y} = \boldsymbol{G}^{\top}(\boldsymbol{x})\,\boldsymbol{z}, \tag{22b}$$

where $\boldsymbol{J}(\boldsymbol{x}) = -\boldsymbol{J}^{\top}(\boldsymbol{x})$ and $\boldsymbol{z}^{\top}\boldsymbol{\mathcal{R}}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) \geq 0$ for all $\boldsymbol{x} \in \mathbb{X}$, $\boldsymbol{z} \in \mathbb{Z}$, $\boldsymbol{u} \in \mathbb{U}$.

Proof. The proof follows directly from Proposition 3.2 under $\boldsymbol{Z}_{\mathrm{RP}}(\boldsymbol{x}) = 0$ and $\boldsymbol{Z}_{\mathrm{PP}}(\boldsymbol{x}) = 0$. □

The results from Proposition 3.2, Proposition 3.3, and Corollary 3.4 are now discussed in the following section.

IV. Discussion

Equation (5) represents an explicit PHS for systems with nonlinear energy-dissipation and feedthrough. The matrices, functions, and vectors in (5) allow for a deep physical insight as they reflect the physical structure of the underlying system. The state $\boldsymbol{x}$ of the system contains the states of the storage elements. The vector $\boldsymbol{z}$ is the co-state of the system and is given by $\boldsymbol{z} = \frac{\partial H}{\partial \boldsymbol{x}}(\boldsymbol{x})$, where the Hamiltonian H is a storage function which describes the total energy contained in the system. The input $\boldsymbol{u}$ and the output $\boldsymbol{y}$ contain the input and output variables, respectively, of the ports of the energy sources. Therewith, the instantaneous power exchange between the system and its environment is given by $\boldsymbol{u}^\top \boldsymbol{y}$. The skew-symmetric matrix $\boldsymbol{J}(\boldsymbol{x})$ represents the internal energy-preserving interconnection in the system. The functions $\boldsymbol{\mathcal{R}}$ and $\boldsymbol{\mathcal{P}}$ account for energy-dissipating effects. In presence of nonlinear energy-dissipating effects, these functions will be also nonlinear. The matrix $\boldsymbol{G}(\boldsymbol{x})$ specifies the interaction between the system and its environment via the system ports. Finally, the matrix $\boldsymbol{M}(\boldsymbol{x})$ is the feedthrough matrix.

The PHS (5) is passive in consequence of its formulation (see Proposition 3.3). Therewith, this formulation is an ideal basis for applying the powerful methods from passivity-based control. For the case of no feedthrough, the system (5) simplifies to the well-known PHS from [9, Def. 6.1.4] (see Corollary 3.4). This verifies the correctness of the port-Hamiltonian formulation from Proposition 3.2.

Proposition 3.2 contains specific calculation rules for the matrices, functions, and vectors in (5). Hence, the determination of such a PHS can be fully automated in a technical computing systems which makes this approach appealing for the modeling of large-scale systems.

Note that Assumption 3.1 is a restriction as it excludes systems with interdependent energy-dissipating elements. On the other hand, this assumption is justified: in the nonlinear case, the presence of interdependent dissipative elements in general disallows to formulate the system in form of an ODE.

V. Conclusion

In this paper, we presented an explicit port-Hamiltonian formulation of systems with nonlinear dissipation and feedthrough (i.e., Proposition 3.2). We provide calculation rules which enable to compute such a PHS in an automated manner based on the constitutive relations of the energy-storing and energy-disspating elements and the Dirac structure of the system. The PHS is proven to be passive (i.e., Proposition 3.3) and generalizes the well-known “PHS with nonlinear resistive structure” from [9, Def. 6.1.4] by feedthrough (i.e., Corollary 3.4). Future work will address the application of this class of systems for the modeling of mechatronic systems and gas networks.

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