Bidirectional travelling surface waves in a convecting fluid

Bidirektionale laufende Oberflächenwellen in einem Konvektionsfluid

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Abstract **—** In the present paper have been obtained exact, solitary, generalized periodic and real periodic solutions of one nonlinear and nonintegrable equation, describing the dynamics of bidirectional surface waves in a convecting fluid. This equation was introduced by M. C. Depassier, and its exact solutions are presented below for the first time.

Zusammenfassung **—** Im vorliegenden Beitrag sind exakte solitäre und verallgemeinerte periodische Lösungen der nichtlinearen Gleichung erhalten, die die Dynamik der bidirektionalen Oberflächenwellen in einem Konvektionsfluid beschreibt. Diese Lösungen werden zum ersten Mal veröffentlicht.

I. Introduction

In 1990 H. Aspe and M. Depassier [1] introduced the evolution equation of unidirectional waves in a convecting fluid

$$u_t + \alpha_0 u u_x + \alpha_3 u_{xxx} + \varepsilon[\alpha_1 (u u_x)_x + \alpha_2 u_{xx} + \alpha_4 u_{xxxx}] = 0,$$

where $u = u(t,x)$ characterizes the surface elevation of the fluid, while $\varepsilon, \alpha_0, \dots, \alpha_4$ are parameters associated with the dynamic state of the fluid, as $0 < \varepsilon \ll 1$. A similar assumption of unidirectionality is substantiated for waves in plasma, in a fluid flow along an inclined plane, but is not justified for a convecting fluid with long waves developing on its surface. That is why M. Depassier [2], based on the Oberbeck-Boussinesq system [3] and equipped with appropriate boundary conditions, presents the following equation:

$$\begin{aligned} &u_{tt} - u_{xx} - \varepsilon^2 (a_0 u_{ttxx} + a_1 u_{xx}^2) + \\ &+\varepsilon^3 (R_0 u_{txx} + b_0 u_{txxxx} + b_1 u_{txx}^2) = 0, \end{aligned} \quad (1)$$

where $a_0 = \frac{21+34Pr}{21Pr^2 Ga}$, $a_1 = \frac{3}{2} + \frac{15}{PrGa}$, $R_0 = \frac{2(Ra-30)}{15\varepsilon^2 Pr\ Ga}$

$$b_0 = \frac{B}{Pr^4 Ga^2},\ b_1 = \frac{8}{PrGa},$$

$B = \frac{2798}{2079} Pr + (GaPr^2 - 1)(\frac{1265}{12096} + \frac{3973}{60480\ Pr})$

where Pr – the Prandtl number, Ga – the Galilean number and Ra – the Rayleigh number.

While there have been no exact localized solutions of equation (1) published so far, a number of authors [4],[5],[6],[7],[8] have obtained various exact solutions of its unidirectional version.

In this work has been applied the mapping and singular deformations method to obtain solitary localized solutions of the bidirectional equation (1), as well as one spatial variation of the Hirota-Satsuma bilinear transformation method [9], described in detail in [8],[10] to obtain a generalized periodic solution.

Equation (1) gives grounds to be considered as third approximation for the small parameter ε of the classical one dimensional wave equation $u_{tt} = u_{xx}$, which is the zero approximation of (1).

II. Mathematical notes

Due to the presence of four terms with mixed partial derivatives in (1), the last one is not invariant with regard to the Galilean transformation, as it does not possess the status of an evolution equation. For convenience in the analysis, let us rescale the variables in equation (1) in the form:

$$t \to \left(\frac{b_1}{a_1}\right) t;\quad x \to \sqrt[4]{\frac{a_1}{b_1 b_0^3}}\, x;\quad u \to \frac{1}{b_1}\sqrt{\frac{a_1}{b_0 b_1}}\, u\,,$$

thus reducing equation (1) to the following

$$\begin{aligned} &u_{tt} - \alpha u_{xx} - \varepsilon^2 [\beta u_{ttxx} + (u^2)_{xx}] + \\ &+\varepsilon^3 (\gamma u_{txx} + u_{txxxx} + (u^2)_{txx}] = 0\,, \end{aligned} \quad (2)$$

where $\alpha = \sqrt{b_1/a_1 b_0}$; $\beta = a_0\sqrt{a_1/b_0 b_1}$; $\gamma = R_0\sqrt{b_1/a_1 b_0}$. Further on, we will analyze the more compact form of equation (2).

III. Localized solitary solutions

Let us assume that the solution of (2) can be repre-sented in the form of a "traveling" wave $u(t,x) = f(\xi)$, where $\xi = kx + \omega t + \delta$, and $f(\xi)$ is an unknown, sufficiently smooth function, and the parameters of the phase k, ω, δ which in the general case can also be complex. But in order to avoid the trivial solution we will assume that $k \neq 0$ and $\omega \neq 0$. Applying this reduction and after double integration with respect to x, equation (2) is reduced to the following nonlinear ordinary differential equation

$$\begin{aligned} &(\omega^2 - \alpha k^2) f - \varepsilon^2 k^2 (\beta\omega^2 f'' + f^2) + \\ &+\varepsilon^3 k^2 \omega (\gamma f' + k^2 f''' + 2 f f') = S, \end{aligned} \quad (3)$$

where $S =$ const, but in the general case $S = S_0\xi + S_1$ and on the assumption that $S_0 = 0$ then the integration constant is real.

We will assume that the correlation

$$f(\xi) = A\varphi(\xi) + E + B\varphi'(\xi)/[\varphi(\xi) - C]\,,$$

represents the solution of the Riccati equation

$$\varphi'(\xi) = \sqrt{4\varphi^3(\xi) - 12C^2\varphi(\xi) + 8C^3}\ ,$$

in the initial equation (3). We will assume that the unknown at this stage parameters A, B, C, E, S are such that $A^2 + B^2 \neq 0$, in order to avoid the trivial solution, and $C \in \mathbb{R}$ withal $C > 0$. Under the hypothesis for the Riccati equation, we have (See [11],[12])

$$\varphi(\xi) = \wp(\xi, 12C^2, -8C^3) ,$$

where $\wp(\xi, g_2, g_3)$ is the Weierstrass elliptic function [12] with real invariants $g_2 = 12C^2$, $g_3 = -8C^3$ for $C > 0$. Substituting $f(\xi)$ with its equal in the initial equation (3) we obtain the following algebraic system for the coefficients in front of $\wp$ and $\wp'$.

$$\begin{cases} A(A + 6k^2) = 0 \\ 2B(\beta\omega^2 + A) = \varepsilon\omega(\gamma A + 2AE + 4B^2) \\ B[(\omega^2 - \alpha k^2) - 2\varepsilon^2 k^2(AC + E)] = \\ = 18\varepsilon^3 k^2 \omega AC^2(A + 6k^2) \\ A(A + 6\beta\omega^2) = 12\varepsilon\omega B(A + k^2) \\ A(\omega^2 - \alpha k^2) = 2\varepsilon^2 k^2[AE + 2B^2 - \varepsilon\omega B(2AC + 2E + \gamma)] \\ S = -2BC\{(\omega^2 - \alpha k^2) + 2\varepsilon^2 k^2[C(3\lambda\omega^2 + 4A) + E]\} + \\ +\varepsilon^2 k^2 \omega C^2\{6A(16k^2 C - \gamma) + 4[A(4AC - 3E) - 6B^2]\} \end{cases} \quad (4)$$

Interesting from the physical point of view are only those real solutions of system (4) for which $A^2 + B^2 \neq 0$, $\omega \neq 0$, $C > 0$ for a wave number $k > 0$, which we accept as a parameter. This system admits two families of such solutions – for $B \neq 0$ and for $B = 0$. If $B \neq 0$, then the system admits the solution:

$$\begin{cases} A = -6k^2; \; B = \dfrac{3(\beta\omega^2 - k^2)}{5\varepsilon\omega} \\ E = \dfrac{(\beta\omega^2 - k^2)(\beta\omega^2 + 24k^2)}{5\varepsilon^2 k^2 \omega^2} - \dfrac{\gamma}{2} \\ C = \dfrac{[(\beta^2 - 25)\omega^4 + k^2(25\alpha + 23\beta - 25\gamma\varepsilon^2)\omega^2 - 24k^4]}{300\varepsilon^2\omega^2 k^2} \\ S = 72\varepsilon^3 k^2 \omega C^2[k^2(3\gamma + 4E) + 2B^2] - \\ -2BC\{(\omega^2 - \alpha^2 k^2) + 2\varepsilon^2 k^2[3C(\beta\omega^2 - 8k^2) + E]\} \end{cases} \quad (5)$$

and the characteristic equation for th frequency ω is

$$\lambda^3 - k^2\left(\frac{6}{\beta} + \alpha + \beta - \gamma\varepsilon^2\right)\lambda^2 + k^4(\alpha + \gamma\varepsilon^2)\lambda - \frac{6k^2}{\beta} = 0 \quad (6)$$

as $\lambda = \omega^2$. To be correct we need to study for which values of the apriori parameters $\varepsilon, \alpha, \beta, \gamma$ and $k > 0$, the parameter C from the solution (5) is positive, i.e. $C > 0$, under the condition that $0 < \varepsilon \ll 1$; $\alpha > 0$; $\beta > 0$; $\gamma > 0$. If we denote the function

$$\psi(\lambda) = (\beta^2 - 25)\lambda^2 + k^2(23\beta + 25\alpha - 25\gamma\varepsilon^2)\lambda - 24k^4$$

then for $\beta > 5$ this function is convex and has only one positive root λ_2, i.e. $C(\omega) > 0$. If $0 < \beta < 5$ then $\psi(\lambda)$ assumes positive values if $0 < \omega^2 < \lambda_2$ since $\psi(\lambda)$ is concave. The above conditions which hold for every $k > 0, \alpha > 0$ and $\varepsilon \in (0,1)$, provide $C(\omega) > 0$. Finally, if $\beta = 5$, then $\psi(\lambda) > 0$ only if

$$\omega^2 = \lambda > \frac{24k^2}{5(23 + 5\alpha - 5\gamma\varepsilon^2)}$$

The above conditions providing $C > 0$, have a meaning only if the characteristic equation (6) allows at least one positive root. That's really the case, since the signs of the variations of its coefficients is either one or three, no matter what the sign of its second coefficient is and according to the Descartes theorem, this equation has either three positive roots or one triple positive root. Taking into account the correlations for the elliptic functions with invariants $g_2 = 12C^2$, $g_3 = -8C^3$ for $C > 0$, defined by (5)

$$\wp(\xi, g_2, g_3) = 3C\coth^2(\xi\sqrt{3C}) - 2C$$
$$\wp'(\xi, g_2, g_3) = (\wp - C)\sqrt{3C}\coth(\xi\sqrt{3C});$$

then we obtain the following family of solitary-wave solutions

$$\begin{aligned} u(t,x) = &-18k^2 C.\coth^2\left(\xi\sqrt{3C}\right) + \\ &+\frac{3\sqrt{3C}(\beta\omega^2 - k^2)}{5\varepsilon\omega}\coth\left(\xi\sqrt{3C}\right) + \\ &+\left[\frac{(\beta\omega^2 - k^2)(\beta\omega^2 + 24k^2)}{50\varepsilon^2 k^2\omega^2} + \frac{24k^2 C - \gamma}{2}\right] \end{aligned} \quad (7)$$

This family of wave solitary solutions, with free parameters $k > 0$ and δ, is structured by two pulses of the type $\coth^r(\xi\sqrt{3C})$, $r = 1, 2$ with equal phases and spatial displacement (in the square brackets). Such waves are called shock waves. If in the system (5) $B = 0$, then it allows the following solution

$$\begin{aligned} &A = -6k^2; \; B = 0; \; E = -\frac{\gamma}{2} \\ &C = \frac{\beta(\beta^2 - 2) + 25(\alpha - \beta - 1) - 24k^4}{300\varepsilon^2} \\ &\varepsilon^2 = \frac{\alpha\beta - 1}{\gamma}; \; \omega = \pm k\sqrt{\beta} \\ &S = \varepsilon^2 k^2(\gamma^2 - 12k^4 g_2). \end{aligned} \quad (8)$$

If in the condition of the hypothesis about the real invariants $g_2 = 12C^2$, $g_3 = -8C^3$ with $C > 0$, we apply the phase modulation $\delta = iK_2/\sqrt{e_1 - e_3}$, where $e_1 = e_2 = C, e_3 = -2C$, then (See [11],[12])

$$\wp(\xi, 12C^2, -8C^3) = C - 3C\,\mathrm{sech}^2(\xi\sqrt{3C}),$$

where $\xi \to kx + \omega t$. The following solitary-wave solution is generated:

$$u(t,x) = 18k^2 C.\mathrm{sech}^2\left(\xi\sqrt{3C}\right) - (6k^2 C + \gamma/2) \quad (9)$$

and all the parameters in this solution are defined by the equality (8). The above solitary solution cannot be degenerated into a soliton one since for $k^2 = \gamma/12C$. The requirement $C > 0$ according to (8) means that

$$0 < k < \{[\beta(\beta^2 - 2) + 25(\alpha - \beta - 1)]/24\}^{1/4} ,$$

which is an additional restriction for the wave number.

IV. Periodic solutions

Let us apply the Hirota-Satsuma transformation [9] to the solution of equation (2)

$$u(t,x) = c + \mu(\ln\zeta)_{xx} , \quad (10)$$

where c, μ are unknown parameters at this stage, and let us assume that $\mu \neq 0$ in order to avoid the trivial solution, while $\zeta = \zeta(t,x)$ belongs to the class of functions $W_F^{2,6}(\Omega)$ containing all the real and complex functions defined in the stripe $\Omega = \{(t,x) \in \mathbb{R}^2, 0 < t < \infty, -\infty < x < \infty\}$ which have continuous derivatives with respect to t and x up to second and sixth order, and this space possesses a strong topology. After substituting $u(t,x)$ from (10) in (2), single integration with respect to x and representing the logarithmic derivatives by the Hirota bilinear operator

$$D_t^n D_x^m \varphi(t,x).\psi(t,x) =$$
$$= (\partial/\partial t - \partial/\partial t')^n (\partial/\partial x - \partial/\partial x')^m \varphi(t,x).\psi(t',x')|_{t=t',x=x'}$$

(See [11, 12]) we will obtain a bilinear reduction of (2)

$$\frac{1}{2\zeta^2}[D_t^2-\alpha D_x^2-\varepsilon^2\beta D_t^2 D_x^2-8G]\zeta.\zeta+$$
$$+\frac{\varepsilon^2}{\zeta^2}\left(\frac{D_x^2\zeta.\zeta}{2\zeta^2}\right)[6\beta D_t^2\zeta.\zeta-\mu D_x^2\zeta.\zeta-2c\zeta^2]+$$
$$+\varepsilon^3\frac{\partial}{\partial t}\left\{\frac{3}{\zeta^2}[D_x^4+(\gamma+2c)D_x^2+8k^2H]\zeta.\zeta\right\}+$$
$$+(\mu-6)\left(\frac{D_x^2\zeta.\zeta}{2\zeta^2}\right)^2=0$$

where $8G$ is a resultant constant of integration, while $8k^2H$ is the so-called "differential" constant whose role will be specified later.

If in the last bilinear equation, we choose $\mu=6$, then it is represented as a conjunction of three residual equations

$$(D_t^2-\alpha D_x^2-\varepsilon^2\beta D_t^2 D_x^2-8G)\zeta.\zeta=0 \quad (11)$$
$$[D_x^4+(\gamma+2c)D_x^2+8k^2H]\zeta.\zeta=0 \quad (12)$$
$$2c\zeta^2=3\beta D_t^2\zeta.\zeta-3D_x^2\zeta.\zeta \quad (13)$$

The presence of three residual equations, one of which (13) not being polynomial-bilinear with respect to the operators D_t and D_x, clearly demonstrates the non-integrable nature of the initial equation (2). If there exists a sufficiently smooth function $\zeta(t,x)$ of the class $W_F(\Omega)$, satisfying all the three residual equations, then the function $u(t,x)$, formed by the ratio (10), would be an exact localized solution of equation (2). For this purpose, we will assume that the Jacobi θ-function (See [12]) is a probable solution

$$\zeta(t,x)=\theta_3(\xi,q)=\sum_{n=-\infty}^{\infty}q^{n^2}e^{2in\xi},$$

where $\xi=kx+\omega_0 t+\delta$ and $q=e^{i\pi\tau}$, $Im\tau>0$ i.e. $0<|q|<1$. This function is a well-defined biperiodic function for $|q|\in(0,1)$ with real period π and quasi period 2. In parallel, we will check the residual equations (11) and (12) as regards their solvability with the function $\theta_3(\xi,q)$ as they have a similar polynomial-bilinear structure. If we substitute $\theta_3(\xi,q)$ in equations (11) and (12) and use the properties of the bilinear operators $D_t^m\zeta.\zeta$, $D_x^n\zeta.\zeta$, $D_t^n D_x^m\zeta.\zeta$ (See [13]), we will obtain accordingly the infinite systems

$$\sum_{m=-\infty}^{\infty}F_1(m)e^{2im\xi}=0,\sum_{m=-\infty}^{\infty}F_2(m)e^{2im\xi}=0,$$

where

$$F_1(m)=\sum_{m=-\infty}^{\infty}[-4\omega_0^2(2n-m)^2+$$
$$+4\alpha k^2(2n-m)^2-16\varepsilon^2\beta k^2(2n-m)^4-8G] \text{ for (11)},$$
$$F_2(m)=\sum_{m=-\infty}^{\infty}[16k^4(2n-m)^4-$$
$$-4k^2(\gamma+2c)(2n-m)^2+8k^2H]q^{n^2+(n-1)^2} \text{ for (12)}.$$

Let us apply the index parity principle [14] to the infinite systems above. This principle means that if in the systems $F_j(m)=0$, $m=0,\pm1,\pm2,\dots$, $j=1,2$ we rearrange the terms $n\to n+1$, then we will easily obtain the chains

$$F_j(m)=F_j(m-2)q^{2(m-1)}=\dots=$$
$$=\begin{cases}F_j(0)q^{\frac{m^2}{2}}, & \text{if } m \text{ is even; } j=1,2\\ F_j(1)q^{(m^2-1)/2}, & \text{if } m \text{ is odd; } j=1,2\end{cases}$$

which means that if we group the terms in the infinite sums $F_j(m)=0$ into even and odd addends, we will obtain the two compact equalities:

$$F_j(0)\theta_3(2\xi,q^2)+F_j(1)q^{-1/2}\theta_2(2\xi,q^2)=0,\ j=1,2$$

where $\theta_2(z,q)$ is the second Jacobi θ-func-tion: $\theta_2(z,q)=\sum_{n=-\infty}^{\infty}q^{(n-1/2)^2}e^{i\xi(2n-1)}$, i.e. the infinite systems are reduced to the two pairs of equations

$$F_1(0)=0\,;\ F_1(1)=0 \text{ and } F_2(0)=0\,;\ F_2(1)=0.$$

If we apply the transformation formulas to these two equations, they will transform into the following two linear non-homogeneous algebraic systems

$$q[\theta_3'+8\varepsilon^2\beta k^2(\theta_3'+q\theta_3'')]\omega_0^2+\theta_3 G=\alpha k^2 q\theta_3'\,;$$
$$q[\theta_2+8\varepsilon^2\beta k^2(\theta_2'+q\theta_2'')]\omega_0^2+\theta_2 G=\alpha k^2 q\theta_2'\,;$$
$$8q(\theta_3'+q\theta_3'')]k^2+\theta_3 H=q(\gamma+2c)\theta_3'\,;$$
$$8q(\theta_2'+q\theta_2'')]k^2+\theta_2 H=q(\gamma+2c)\theta_2'\,,$$

as the first system is with relation to ω_0^2 and G, and refers to (11), while the second one is with relation to k^2 and H, and refers to (12). For convenience we have used the denotations $\theta_j=\theta_j(0,q^2)$ and $\theta_j'=d\theta_j/dq$, $j=2,3$. Both algebraic systems are compatible and determined, which is easily verified by the linear independence of θ_2 and θ_3. The only solutions of these systems are

$$\omega_0(q)=\pm\sqrt{\frac{\alpha k^2}{1+8\beta\varepsilon^2k^2W_0(\theta_2,\theta_3)}}$$
$$G(q)=\frac{8\alpha\beta\varepsilon^2 qW(\theta_2',\theta_3')}{W(\theta_2,\theta_3)[1+8\beta\varepsilon^2k^2W_0(\theta_2,\theta_3)]} \quad (14)$$
$$k^2=\frac{\gamma+2c}{8W_0(\theta_2,\theta_3)};H(q)=\frac{q(\gamma+2c)W(\theta_2',\theta_3')}{W(\theta_2,\theta_3)W_0(\theta_2,\theta_3)} \quad (15)$$

where $W(\theta_2,\theta_3)=\theta_2\theta_3'-\theta_2'\theta_3$ is the Wronskian of the functions θ_2 and θ_3, $W'(\theta_2,\theta_3)=\theta_2\theta_3''-\theta_2''\theta_3$, $W_0(\theta_2,\theta_3)=1+qW'(\theta_2,\theta_3)/W(\theta_2,\theta_3)$.

At this stage of the solution, the active parameters are c,q,δ, which could be complex in the general case. It is not difficult to see from (14) that the integration constant $G(q)$ is different from zero for $|q|\in(0,1), k\neq0$ and for every admissible value of the apriori para-meters $\alpha,\beta,\gamma,\varepsilon$. This condition is necessary for the existence in general of a periodic solution to (2).

The suitability of the index parity principle is inapplicable in trying to satisfy the residual equation (13). Hence, we will use a spatial version of the bilinear-transformation method (See [8],[10]), representing the spatial displacement in the formal series

$$c=6(k^2-\beta\omega_0^2)\sum_{m=-\infty}^{\infty}c_m(q) \quad (16)$$

with unknown terms $c_m(q)$, while the wave number k and the phase frequency ω_0^2 are defined by means of (15) and (14), respectively. If we substitute (16) in equation (13) and apply the Cauchy formula

$$\sum_{\nu=-\infty}^{\infty}a_\nu\sum_{s=-\infty}^{\infty}b_s=\sum_{m,n=-\infty}^{\infty}a_m b_{n-m}=\sum_{m,n=-\infty}^{\infty}a_{n-m}b_m$$

we will obtain the following infinite system

$$c_m\sum_{n=-\infty}^{\infty}q^{(n-m)^2+(n-2m)^2}e^{2im\xi}=$$
$$=\sum_{n=-\infty}^{\infty}(2n-m)^2q^{n^2+(n-m)^2}e^{2im\xi}$$

(we suppose that $k^2\neq\beta\omega_0^2$) from which we can determine

$$c_m(q) = \frac{\sum_{n=-\infty}^{\infty}(2n-m)^2 q^{n^2+(n-m)^2}}{\sum_{n=-\infty}^{\infty} q^{(n-m)^2+(n-2m)^2}}, m = 0, \pm 1, \dots \quad (17)$$

It is not difficult to show that the series $\sum_{n=-\infty}^{\infty} c_m(q)$, with terms defined by (17), is absolutely convergent if $0 < |q| < 1$. Indeed, if we rearrange the terms of this series applying the reduction $n \to n + 2m$, then we will obtain the representation

$$\sum_{m=-\infty}^{\infty} c_m(q) = [-R_0(z,q)/9 - 2imR_1(z,q) + +9m^2 R_2(z,q)]q^{4m^2}, \text{where}$$

$$z = m\pi\tau; R_0(z,q) = \frac{\ddot{\theta}_3(3z,q^2)}{\theta_3(z,q^2)}; R_1(z,q) = \frac{\dot{\theta}_3(3z,q^2)}{\theta_3(z,q^2)};$$

$$R_2(z,q) = \theta_3(3z,q^2)/\theta_3(z,q^2).$$

From the last equality follows the estimate

$$\sum_{m=-\infty}^{\infty} |c_m(q)| \le \frac{1}{9}\sum_{m=-\infty}^{\infty} |R_0||q|^{4m^2} + +2\sum_{m=-\infty}^{\infty} |m||R_1||q|^{4m^2} + 9\sum_{m=-\infty}^{\infty} m^2|R_2||q|^{4m^2}$$

i.e. the series of the absolute values is bounded by the superposition of three absolutely convergent series, since $|R_j(z,q)| \le P_j$, $j = 0,1,2$; $P_j = \text{const}$. Furthermore eve-ry series $\sum_{m=-\infty}^{\infty} |m|^{\nu}|q|^{4m^2}$, $\nu \in \mathbb{N}$, $0 < |q| < 1$ is convergent according to the D'Alembert criterion.

We can already make the conclusion that the function $\zeta(t,x) = \theta_3(\xi,q)$, with parameters calculated from (14) and (15) and c, defined by (18), defines an exact localized complex solution of equation (2) in the for

$$u(t,x) = 6(k^2 - \beta\omega_0^2)\sum_{m=-\infty}^{\infty} c_m(q) + 6k^2 \frac{\partial}{\partial\xi}\left[\frac{\dot{\theta}_3(\xi,q)}{\theta_3(\xi,q)}\right] \quad (18)$$

V. Real periodic solutions

The double poles of the complex meromorphic solution (19) being in the lattice

$$\xi_{mn} = (m + 1/2)\pi + \tau(n + 1/2)\pi, \quad m,n \in \mathbb{Z}$$

can be avoided if we limit the variation of the phase variable ξ. Within the condition of the hypothesis for q to be real, i.e. $\tau = ip, p > 0$, whereat $q = e^{i\pi\tau} = e^{-\pi p} \in (0,1)$, this restriction could be $|Im(\xi)| < \pi p$, i.e. if ξ varies in the horizontal stripe $-\pi p < Im(\xi) < \pi p$. This is also an analyticity condition of the localized exact solution (19).

Practical applicability of this solution within the hypothesis $q = e^{-\pi p}$, $p > 0$ can be achieved for two choices of the wave number k from (15). If $k \in \mathbb{R}$, we can assume that $k > 0$. Taking into account the Fourier representation (See [12]) we obtain the following real restriction of (18)

$$u(t,x) = 6(k^2 - \beta\omega_0^2)\sum_{m=-\infty}^{\infty} [c_m(q) + + \frac{2k^2}{k^2 - \beta\omega_0^2}(-1)^m m\text{cosech}(\pi m p).\cos(2m\xi)], \quad (19)$$

which describes a family of sinusoidal harmonics with different amplitudes and individual spatial displacements, developing in zones with weak nonlinearity ($p \to 0$, i.e. $q \to 1$).

If k is an imaginary number (with positive imaginary part), then for suitable choice of the phase displace-ment $\delta \to i\delta$, we can achieve a completely imaginary phase $\xi \to i\xi + i\pi p$ and then, using the quasi-periodic modulation for $\theta_3(z + i\pi p/2, q)$, we will obtain from (18) the periodic solitary-wave solution of (2)

$$u(t,x) = 6(k^2 - \beta\omega_0^2)\sum_{m=-\infty}^{\infty} [c_m(q) + + \frac{k^2}{k^2 - \beta\omega_0^2}\text{sech}^2(\xi - m\pi p)] \quad (20)$$

VI. Conclusions

In this periodic real solution is carried out a complex superposition of sech^2-profiles, each of which having its own spatial displacement $c_m(q)$. In this case we cannot talk about the existence of one solitary wave which would be an envelope of the solitary wave profiles (21), because the phase velocity ω/k of the solitary wave (7), cannot become equal to the phase velocity ω_0/k of the periodic wave (21), which is valid in the zones of strong nonlinearity ($p \to \infty$, i.e.$q \to 0$).

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